Geometric Group Theory
(Final Test, BMath-3rd year, 2026)

Instructions: Total time 3 Hour. Solve any **FIVE** problems. Use terminologies, notations and results as covered in the course, no need to prove such results. If you wish to use a problem given in a homework or an exercise from the class, please sketch a solution.

1. Let G and H be finitely generated infinite groups. Prove that the group $G \times H$ has exactly one end. (10)

2. Prove that the cyclic group $\mathbb{Z}$ is not quasi isometric to the free abelian group $\mathbb{Z}^n$ for any $n \geq 2$. (10)

3. Prove that $\mathbb{R}^m \sim \mathbb{R}^n$ if and only if $m = n$. (10)

4. Let G be a group with a finite set of generators S. Let $\beta : \mathbb{N} \to \mathbb{N}$ be the growth function defined by

   $\beta(n) = |B(e, n)|$, the number of elements in the closed n-ball $B(e, n)$ around the identity element e in the metric space (G, d_S), where d_S denotes the word metric on G with respect to S. Prove
   (i) $\beta(m + n) \leq \beta(m)\beta(n)$ for all $m, n \in \mathbb{N}$.
   (ii) Let N be a normal subgroup of G. Prove that $\beta_{G/N} \leq \beta_G$, where subscripts distinguish the respective growth functions. (5+5)

5. What can you say about the growth type (i.e. polynomial, exponential etc) of the group $SL(2, \mathbb{Z})$? Explain your answer. (10)

6. Let $G = SL(2, \mathbb{Z})$ and $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let H be the centralizer of x in G. Prove that H is virtually cyclic. Find the index of $< x >$ in H. (5+5)

7. Let G be a hyperbolic group and let $x \in G$ have infinite order. Prove that, for any $y \in G$ with $xy = yx$, the subgroup $< x, y >$ of G generated by x, y, is virtually cyclic. (10)

8. Prove that a finitely generated abelian group G is hyperbolic iff either $e(G) = 0$ or $e(G) = 2$. (10)